

AE-802

M.A./M.Sc. (Previous)
Term End Examination, 2016-17

MATHEMATICS

Compulsory

Paper - IV

Complex Analysis

Time : Three Hours] [*Maximum Marks* : 100

[*Minimum Pass Marks* : 36

Note : Answer any **five** questions. Answer to each question should begin on a fresh page. All questions carry equal marks.

1. (a) If a function $f(z)$ is analytic within and on a closed contour C and a is any point lying in it, then prove that

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

- (b) Obtain the Taylor's series and Laurent series which represent the function

$$\frac{z^2 - 1}{(z+2)(z+3)} \text{ in the regions—}$$

(i) $2 < |z| < 3$ (ii) $|z| > 3$

(iii) $|z| < 2$

(2)

2. (a) State and prove Morera's theorem.

(b) Prove that the function $\sin \left\{ c \left(z + \frac{1}{z} \right) \right\}$ can be expanded in a series of the type

$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ in which the coefficients of both z^n and z^{-n} , are

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(2c \cos \theta) \cos n\theta d\theta$$

3. (a) Find the residues at all singularities in a

contour C of the functions $\frac{e^z}{z^3 - z^2}$ and $\frac{z+1}{z^2(z-3)}$.

(b) By the method of contour integration, prove that

$$\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$$

4. (a) Find all the Mobius transformations which transform the half plane $I(z) \geq 0$ into circles $|w| \leq 1$.

(b) If $w = f(z)$ represents a conformal mapping of a domain D in the z -plane into a domain D' of the w -plane, then prove that $f(z)$ is an analytic function of z in D .

(3)

5. (a) State and prove Montel's theorem.
(b) Prove that the space of analytic functions $H(G)$ is complete metric space.
6. (a) For $0 < \operatorname{Re} z < 1$, prove that

$$\zeta(z)\sqrt{z} = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} dt$$

- (b) Define the following terms :
- (i) Analytic continuation along a chain of domains
 - (ii) Analytic continuation along a curve
 - (iii) Complete analytic function
 - (iv) Natural boundary
 - (v) Function element
7. (a) Let f be a function defined on $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$. Suppose that f has the following properties :
- (i) $\log f(x)$ is a convex function
 - (ii) $f(x+1) = xf(x)$ for all x
 - (iii) $f(1) = 1$

Then prove that $f(x) = \sqrt{x}$ for all x .

- (b) State and prove Monodromy theorem.
8. (a) Suppose $f(z)$ is analytic in a closed ring $r_1 \leq |z| \leq r_3$. Let $r_1 < r_2 < r_3$ and $M(r_i)$ be the maximum value of $|f(z)|$ on the circles $|z| = r_i$ ($i = 1, 2, 3$). Then prove that

(4)

$$\log M(r_2) \leq \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} \log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \log M(r_3).$$

(b) State and prove Jensen's inequality.

9. (a) Let f be a non-constant entire function of order λ with $f(0) = 1$ and let $\{a_1, a_2, \dots\}$ be the zeroes of f counted according to multiplicity and arranged so that

$$|a_1| \leq |a_2| \leq \dots$$

If an integer $p > \lambda - 1$, then prove that

$$\frac{d^p}{dz^p} \left[\frac{f'(z)}{f(z)} \right] = -p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}} \text{ for}$$

$$z \neq a_1, a_2, \dots$$

- (b) If $|z| \leq 1$ and $p \geq 0$, then prove that

$$|1 - E_p(z)| \leq |z|^{p+1}$$

where $E_p(z)$ are elementary factors of an analytic function.

10. (a) State and prove Little Picard's theorem.

- (b) Suppose g is analytic on $B(0, R)$, $g(0)=0$, $|g'(0)| = \mu > 0$ and $|g(z)| \leq M$ for all z , then prove that

$$g(B(0, R)) \supseteq B\left(0, \frac{R^2 \mu^2}{6M}\right)$$