

# AE-805

M.A./M.Sc. (Final)  
Term End Examination, 2016-17

# MATHEMATICS

## Compulsory

# Paper - I

# Integration Theory and Functional Analysis

*Time : Three Hours]                      [Maximum Marks : 100*  
*[Minimum Pass Marks : 36*

**Note** : Answer any **five** questions. All questions carry equal marks.

1. (a) State and prove Extension theorem.  
(b) State and prove Hahn decomposition theorem.
2. (a) Explain the Baire sets and Baire measure.

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(b) Let  $(X, A, \mu)$  be finite measure space, let

$p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $F$  is a continuous

linear function of  $L_p(X)$ , then there exists

$a, g \in L_q(X)$  such that

$$F(f) = \int_X fg \, d\mu, \text{ for all } f \in L_p(X)$$

and  $\|F\| = \|g\|_q$ . Prove it.

3. (a) Show that  $l_\infty, C, C_0$  are nls each with the norm  $\|x\| = \sup |x_n|$ .

Does  $\|x\| = \lim_{n \rightarrow \infty} |x_n|$  define a norm on  $C$ ?

(b) A nls  $X$  is complete if and only if every absolutely convergent series in  $X$  is convergent.

4. (a) Show that two equivalent norms on a linear space  $X$  induce the same topology on  $X$ .

(b) State and prove Fubini's theorem.

5. (a) State and prove Borel-Lebesgue theorem.

(b) Let  $\{x_n\}$  be a weakly convergent sequence in a normed space  $X$ , i.e.  $x_n \xrightarrow{w} x$ . Then prove that

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- (i) Weak limit of the sequence  $\{x_n\}$  is unique.
  - (ii) Every subsequence of  $\{x_n\}$  convergence weakly to  $x$ .
  - (iii) The sequence  $\|x_n\|$  is bounded.
6. (a) State and prove Hahn-Banach theorem for real linear spaces.
- (b) Show that the Schwarz inequality in Hilbert space.
7. (a) Explain the Gram-Schmidt orthogonalization process.
- (b) Let  $\{e_1, e_2, \dots, e_n\}$  be a finite orthonormal set in a Hilbert space  $H$  and  $x$  be any element of  $H$ . Then prove that

$$\sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2$$

8. (a) If  $T \in B(x, y)$ , then prove that
- $$T^* \in B(y^*, x^*) \text{ and } \|T\| = \|T^*\|.$$
- (b) Let  $X$  and  $Y$  be normed spaces. Then prove that
- (i) Every compact linear operator  $T: X \rightarrow Y$  is bounded, hence continuous.

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(ii) If  $X$  is infinite dimensional space, the identity operator  $I : X \rightarrow X$  (which is continuous) is not compact.

9. (a) State and prove open mapping theorem.
- (b) Let  $T$  be a closed linear map of a Banach space  $X$  into a Banach space  $Y$ . Then  $T$  is continuous. Prove it.
10. (a) State and prove closed range theorem.
- (b) Show that every convergent sequence in a normed linear space is a Cauchy sequence, but the converse need not be true.
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