

AE-800

M.A./M.Sc. (Previous)
Term End Examination, 2016-17

MATHEMATICS

Compulsory

Paper - II

Real Analysis and Measure Theory

Time : Three Hours] [Maximum Marks : 100
[Minimum Pass Marks : 36

Note : Answer any **five** questions. All questions carry equal marks.

1. (a) State and prove the fundamental theorem of calculus.
- (b) Let f be a bounded function and α be a monotonically increasing function on $[a, b]$. Then $f \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

Prove it.

(2)

2. (a) Let f be a continuous function defined on $[a, b]$. Then prove that there exists a sequence of polynomials which converges uniformly to f on $[a, b]$.
- (b) State and prove Tauber's theorem on power series.

3. (a) Show that the series
$$1 - x + x^2 - x^3 + \dots \quad (0 \leq x \leq 1)$$
admits of integration term-by-term in $0 \leq x \leq 1$ although it is not uniformly convergent in $[0, 1]$.

- (b) Show that

$$\int_a^b f \, dx = \lim_{n \rightarrow \infty} \int_a^b f_n \, dx$$

4. State and prove the inverse function theorem.
5. (a) Show that the power series $1 + 2x + 3x^2 + 4x^3 + \dots$ has radius of convergence equal to 1.

- (b) If a power series $\sum a_n x^n$ diverges for $x = x_0$, then prove that it diverges for every $x = x_1$ when $|x_1| < |x_0|$.

(3)

6. (a) Find the radius of convergence of the series

$$\frac{1}{2}x + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \dots$$

- (b) If the series $\sum_{n=0}^{\infty} a_n$ is convergent and has the sum S , then prove that the series

$$\sum_{n=0}^{\infty} a_n x^n \text{ is uniformly convergent for}$$

$$0 \leq x \leq 1 \text{ and } \lim_{x \rightarrow 1} \sum_{n=0}^{\infty} a_n x^n = S.$$

7. (a) Prove that the intersection of a finite number of measurable sets is measurable.
- (b) Prove that the interval (a, ∞) is measurable.
8. (a) Show that a Borel measurable set is Lebesgue measurable.
- (b) If E_1 and E_2 are any measurable sets, then prove that

$$\begin{aligned} m(E_1 \cup E_2) + m(E_1 \cap E_2) \\ = m(E_1) + m(E_2). \end{aligned}$$

(4)

9. (a) Find the rectangular parallelopiped of surface area a^2 and maximum volume.

(b) Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$. Then prove that it is Lebesgue integrable on $[a, b]$ and

$$R\int_a^b f(x)dx = \int_a^b f(x)dx$$

10. (a) Let ϕ be a convex function on $(-\infty, \infty)$ and f an integrable function on $[0, 1]$.

Then $\int_0^1 \phi(f(t))dt \geq \phi\left[\int_0^1 f(t)dt\right]$. Prove it.

(b) State and prove Holder's Inequality.
